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Trapped modes in two-dimensional waveguides

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A two-dimensional acoustical waveguide described by two infinite parallel lines a distance $2d$ apart has a circle of radius $a < d$ positioned symmetrically between them. The potential satisfies the two-dimensional Helmholtz equation in the fluid region between the circle and the lines, and the normal gradient of the potential vanishes on both. For motions which are antisymmetric about the centreline of the guide there exists a cutoff frequency below which no propagation down the guide is possible. It is proved that for a circle of sufficiently small radius there exists a trapped mode, having a frequency close to the cutoff frequency, which is antisymmetric about the centreline of the guide and symmetric about a line through the centre of the circle perpendicular to the centreline. The method used is due to Ursell (1951) who established the existence of a trapped surface wave mode in the vicinity of a long totally submerged horizontal circular cylinder of small radius in deep water. Numerical computations in the present work reveal that a single trapped mode appears to exist for all values of $a \leqslant d$ and not just when the circle is small. The present method, when used to attempt to construct a solution antisymmetric about both the centreline and a line perpendicular to it through the centre of the circle does not lead to a trapped mode. The trapped modes can equally well be regarded as surface-wave modes, as in an infinitely long tank of water with a free surface, into which has been placed symmetrically, a vertical rigid circular cylinder extending throughout the depth. Numerical evidence for the existence of such trapped modes when the cylinder is of rectangular cross-section was presented in Evans & Linton (1991).

1. Introduction

The existence of trapped modes in linearized water-wave theory was first established by Ursell (1951). He proved that modes of oscillation which have a frequency below a certain cutoff frequency exist in the vicinity of an infinitely long submerged horizontal cylinder of sufficiently small radius, in deep water. In a highly mathematical paper, Jones (1953), using results from the theory of unbounded operators, generalized Ursell's results to cylinders of arbitrary but symmetric cross-section, and finite water depth. More recently Ursell (1987) has provided a simplified proof of Jones' results using minimum-energy arguments.

Less is known about trapped modes in linear acoustics where the governing equation is the Helmholtz equation although the major part of the paper by Jones (1953) is, in fact, devoted to this equation. Numerical evidence for their existence is provided by recent work of Evans & Linton (1991) who computed the frequencies of trapped modes which occur in an open water channel containing a symmetrically placed rectangular block extending throughout the water depth. These trapped modes, which do not appear to have been discovered previously, are antisymmetric about the centreplane of the channel, and have a wave frequency depending upon the dimensions of the block relative to the channel, being below the so-called cutoff frequency for the channel. The modes are characterized by having finite total energy, being restricted to the vicinity of the block and decaying rapidly to zero with distance down the channel. The number of such modes increases as the length of the block increases; for a block of length less than the channel width there is just a single mode.

McIver (1991) has used matched asymptotic expansions to derive a general expression for the trapped mode frequency above submerged horizontal cylinders of arbitrary but small cross-section which agrees with the Ursell (1951) result for the submerged small circular cylinder, and has also derived the trapped mode frequency for antisymmetric trapped modes in the vicinity of a vertical cylinder of arbitrary but small cross-section in a water channel of infinite extent. Evans & McIver (1991) have derived similar results for the trapped mode frequency close to cutoff, under the assumption that the bodies were thin and symmetric. In both the vertical cylinder problems considered by McIver (1991) and Evans & McIver (1991) and the problem considered by Evans & Linton (1991), the depth dependence can be separated out and the problems reduce to an acoustic problem for the solution of the two-dimensional Helmholtz equation in a waveguide described by two parallel lines enclosing a symmetrically placed cylindrical section, with vanishing normal gradient on all rigid boundaries.

Trapped modes in acoustics are of considerable interest in providing an example of the non-uniqueness of the boundary-value problem arising when the cylindrical section makes simple harmonic oscillations, which are antisymmetric about the centreline of the waveguide, at the trapped mode frequency. Since neither the treatment of Evans & McIver (1991), Evans & Linton (1991) nor that of McIver (1991) is fully rigorous, nor do such problems appear to be covered by Jones' (1953) general theory, it is desirable that a completely rigorous proof of the existence of trapped modes in acoustic waveguides be presented in a specific case.

The problem to be considered here is probably the simplest configuration for which the method of proof to be used is likely to be successful. A two-dimensional acoustic waveguide is represented by two parallel lines $y = \pm d$, $-\infty < x < \infty$ enclosing a circle $x^2+y^2 = a^2$ with radius $a < d$. A potential ϕ is sought satisfying Helmholtz's equation in the fluid region and having vanishing normal gradient on the circle and the parallel lines. The potential is assumed to be odd in y so that it vanishes on $y = 0$ thus ensuring that a cutoff frequency exists below which no propagation down the guide is possible, and even in x.

The method of construction is precisely that used by Ursell (1951) in proving the existence of trapped surface waves above a small submerged horizontal cylinder. Specifically, multipole potentials, odd in y, even in x, are constructed which, for frequencies less than the cutoff frequency, do not radiate energy down the waveguide. These potentials, each of which satisfies all conditions except that on the circle itself, are singular at the centre of the circle but not in the fluid region. The trapped mode potential is then constructed from a linear combination of all possible multipoles. Application of the circle boundary condition results in a homogeneous infinite system of equations for the unknown coefficients in the multipole expansion of the trapped mode potential. It is proved that the elements of the infinite matrix of this system are such as to guarantee that the system behaves in all respects like a finite system, under certain geometrical conditions on a, d. In particular the determinant Δ_N of the truncated $N \times N$ system tends uniformly to the infinite determinant Δ_∞ as $N \to \infty$. All that remains to prove the existence of a trapped mode is to show that Δ_∞ vanishes for some real values of the parameters of the problem and that for these values the trapped mode potential does not vanish identically. In a manner which follows closely that used by Ursell (1951) it is proved that for sufficiently small cylinders close to the cutoff frequency the infinite determinant does indeed vanish when a certain relation between the parameters is satisfied. Computations of Δ_N for increasing N also indicate that Δ_∞ vanishes just once for all real $a \leqslant d$.

It is possible to use the same construction to seek trapped modes which are odd in y and also odd in x, but it turns out that this does not lead to trapped modes for any values of a/d.

2. Formulation

We seek a potential $\phi(x,y)$ satisfying

$$(\nabla^2+k^2)\phi = 0 \quad \text{in} \quad r > a, \quad |y| < d, \quad r = (x^2+y^2)^{\frac{1}{2}}, \tag{2.1}$$

$$\phi_y = 0, \qquad |y| = d, \; -\infty < x < \infty \tag{2.2}$$

$$\phi_r = 0, \qquad r = a, \tag{2.3}$$

$$\phi = 0, \qquad y = 0, \quad |x| \geqslant a, \tag{2.4}$$

$$\phi \to 0, \qquad |x| \to \infty, \quad |y| \leqslant d. \tag{2.5}$$

Thus ϕ can be regarded as a time-independent acoustic potential, the actual potential being derived from $\mathrm{Re}\,\phi \exp(\mathrm{i}\omega t)$ where $k = \omega/c$ and c is the velocity of sound. Equivalently as in Evans & Linton (1991) the equations describe a water-wave problem in which a vertical cylinder extends throughout the depth H thereby permitting a depth dependence $\cosh k(z+H)$ to be separated out from the governing Laplace's equation. In this case k is the positive real root of

$$\omega^2 = gk \tanh kH.$$

Note that the crucial necessary condition for a trapped mode is condition (2.4) since, provided $k < \pi/2d$, no wave radiation to $x = \pm\infty$ is possible. We shall impose the further condition

$$\phi_x = 0, \quad x = 0, \quad a < y < d, \tag{2.6}$$

so that ϕ is even in x and we need only seek ϕ in $x \geqslant 0$, $0 \leqslant y \leqslant d$, $r \geqslant a$, with (2.4), (2.6) providing the extension of ϕ to the larger fluid region.

In the absence of the channel walls a fundamental multipole which satisfies (2.1) and (2.4) is

$$H_n^{(1)}(kr) \sin n\theta, \tag{2.7}$$

where $H_n^{(1)} = J_n + \mathrm{i}Y_n$, $x = r\cos\theta$, $y = r\sin\theta$ and $H_n^{(1)}$, J_n, Y_n are the usual Bessel functions as defined in, for example Watson (1966). For $n = 2m+1$, m integer, this is symmetric about $x = 0$ and satisfies (2.6), whilst for $n = 2m$, m integer, we have antisymmetry about $x = 0$. The corresponding expression with $\sin n\theta$ replaced by $\cos n\theta$ is even in y and even/odd in x according as to whether n is even/odd. The expression (2.7) with $n = 2m+1$ will be the building block for constructing trapped modes. We first need to add image terms at $y = \pm 2jd$, $j = 1, 2, \ldots$, to ensure that (2.2) is satisfied. It is sufficient to consider $0 \leqslant y \leqslant d$ only and choose functions odd in y

to satisfy (2.2) on $y = -d$ and (2.4). It is shown in Appendix A that the suitably modified multipole is

$$\psi_{2n+1}(r,\theta) = Y_{2n+1}(kr)\sin(2n+1)\theta + \mathrm{Re}\frac{(-1)^n}{\pi}\int_{-\infty}^{\infty+\mathrm{i}\pi}\frac{\mathrm{e}^{\gamma d}}{\cosh\gamma d}\sinh\gamma y\cos(kx\cosh v)\,\mathrm{e}^{-(2n+1)v}\,\mathrm{d}v, \quad (2.8)$$

where $\gamma = k\sinh v$ and where the contour is taken along the negative real axis, up to the imaginary axis to $\mathrm{i}\pi$ and then along the line $\mathrm{i}\pi + s$, $s > 0$. The expression (2.8) satisfies (2.1), except at $r = 0$, (2.2) for $y = d$ from (A 3), (2.4), (2.6) and, from (A 6), (2.5). It is also shown in Appendix A to have the expansion

$$\psi_{2n+1}(r,\theta) = Y_{2n+1}(kr)\sin(2n+1)\theta + \sum_{m=0}^{\infty} A_{mn}J_{2m+1}(kr)\sin(2m+1)\theta, \quad (2.9)$$

valid for $r > 0$, where

$$A_{mn} = \frac{-4}{\pi}(-1)^{m+n}\int_0^{\infty}\frac{\mathrm{e}^{-\gamma d}\sinh(2n+1)v\sinh(2m+1)v}{\cosh(\gamma d)}\mathrm{d}v - \frac{4}{\pi}\int_0^{\frac{1}{2}\pi}\tan(\beta d)\cos(2n+1)u\cos(2m+1)u\,\mathrm{d}u, \quad (2.10)$$

where $$\beta = k\cos u.$$

We now seek a trapped mode solution in the form

$$\phi(r,\theta) = \sum_{n=0}^{\infty} k^{-1}a_n(Y'_{2n+1}(ka))^{-1}\psi_{2n+1}(r,\theta), \quad (2.11)$$

where the dash refers to the derivative with respect to the argument. Application of the cylinder condition (2.3) to (2.11) gives, using (2.9)

$$\sum_{n=0}^{\infty} a_n\left\{\sin(2n+1)\theta + \sum_{m=0}^{\infty} A_{mn}\frac{J'_{2m+1}(ka)}{Y'_{2n+1}(ka)}\sin(2m+1)\theta\right\} = 0 \quad (0 \leqslant \theta \leqslant \tfrac{1}{2}\pi), \quad (2.12)$$

whence multiplication by $\sin(2m+1)\theta$ and integration over $[0,\frac{1}{2}\pi]$ results in the homogeneous infinite system of equations

$$a_m + \sum_{n=0}^{\infty} B_{mn}a_n = 0 \quad (m = 0,1,2,\ldots), \quad (2.13)$$

where $$B_{mn} = A_{mn}\frac{J'_{2m+1}(ka)}{Y'_{2n+1}(ka)}. \quad (2.14)$$

It is shown in Appendix B that

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}|B_{mn}| < \infty \quad \text{for } 0 < ka < kd < \tfrac{1}{2}\pi, \quad \coth\chi < M(ka)^{-2}, \quad (2.15)$$

where $kd\cosh\chi = \frac{1}{2}\pi$, and M is independent of ka, kd, which is sufficient to ensure that the determinant Δ_N of the truncated system

$$a_m + \sum_{n=0}^{N} B_{mn}a_n = 0 \quad (m = 0,1,\ldots,N), \quad (2.16)$$

converges uniformly to a limit Δ_∞ (Ursell 1951, p. 357). We can also conclude that the system (2.13) behaves in every respect like a finite system and in particular has a non-trivial solution with $\Sigma|a_m| < \infty$ if and only if the infinite determinant

$$\Delta_\infty = \det(\delta_{mn} + B_{mn}(ka, kd)) \tag{2.17}$$

vanishes for some ka, kd, with $0 < ka < kd < \frac{1}{2}\pi$ and $\coth\chi < M(ka)^{-2}$.

Now the behaviour of the infinite system as $ka \to 0$, kd fixed, is governed by the behaviour of the Bessel functions in the definition of B_{mn} given by (2.14) since A_{mn} is independent of ka. We have from (C 2) in Appendix C

$$\frac{J'_{2m+1}(ka)}{Y'_{2n+1}(ka)} \sim \frac{\pi(2m+1)}{(2n+1)(2n+1)!}(\tfrac{1}{2}ka)^{2n+2m+2} \quad (ka \to 0), \tag{2.18}$$

so that $B_{mn} \to 0$, all m, n as $ka \to 0$, and the only possible solution of (2.13) in this limit is $a_m = 0$. If however $ka \to 0$ and $kd \to \frac{1}{2}\pi$ *simultaneously*, it turns out to be possible to obtain a trapped mode solution similar to that obtained by Ursell (1951) in the horizontal submerged cylinder case. Thus it is shown in Appendix C that the first integral in the definition of A_{mn} given by (2.10) is bounded as $kd \to \frac{1}{2}\pi$ and the second is not. From (C 4) we have

$$A_{mn} \sim -8\pi^{-1}\coth\chi \quad \text{as } \chi \to 0, \tag{2.19}$$

where

$$kd\cosh\chi = \tfrac{1}{2}\pi. \tag{2.20}$$

Thus

$$B_{mn} \sim \frac{-8(2m+1)}{(2n+1)(2n+1)!}\frac{(\frac{1}{2}ka)^{2n+2m+2}}{\tanh\chi}, \tag{2.21}$$

as $ka \to 0$, $\chi \to 0$.

It follows that if $(ka)^2/\tanh\chi = O(1)$ as ka, $\chi \to 0$, then all elements of B_{mn} except B_{00} tend to zero as ka, $\chi \to 0$.

Let

$$(ka)^2 = \tfrac{1}{2}\lambda\tanh\chi, \tag{2.22}$$

for some fixed $\lambda > 0$. Then

$$B_{mn} \sim \frac{-(2m+1)\lambda(ka)^{2n+2m}}{(2n+1)(2n+1)!} \quad (ka \to 0), \tag{2.23}$$

and, since Δ_∞ is uniformly convergent

$$\begin{aligned}\Delta_\infty(ka,\lambda) &= \det(\delta_{mn} + B_{mn}(ka,\lambda)) \\ &\to \det(\delta_{mn} + \lim_{ka\to 0} B_{mn}(ka,\lambda)) \\ &= 1-\lambda.\end{aligned} \tag{2.24}$$

It follows from (2.24) that there exists a range of ka, say $0 < ka < \delta$, for which

$$\Delta_\infty(ka, \tfrac{1}{2}) > 0, \quad \Delta_\infty(ka, \tfrac{3}{2}) < 0.$$

Fix $ka > 0$ in this range. Then $\Delta_\infty(ka,\lambda)$ passes continuously from a negative value to a positive value as λ increases from $\frac{1}{2}$ to $\frac{3}{2}$. In particular there must exist a value of $\lambda = \lambda_0(ka)$ for which

$$\Delta_\infty(ka, \lambda_0(ka)) = 0. \tag{2.25}$$

It is also clear from (2.24) that $\lambda_0 \sim 1$ as $ka \to 0$ so that from (2.22)

$$2(ka)^2 \sim \tanh\chi \sim \chi, \tag{2.26}$$

in agreement with McIver (1991) to this order.

For $\lambda = \lambda_0$ then, there exists a non-trivial solution $\{a_m\}$ of (2.13) with $\Sigma|a_m| < \infty$, and the expression (2.11) describes a trapped mode provided it does not vanish identically and provided $\coth\chi < M(ka)^{-2}$ when (2.26) is satisfied which is clearly the case. Furthermore it is shown in Appendix D that $\phi \neq 0$ and that the expansion (2.11) for ϕ converges like $\Sigma|a_n|/(2n+1)$. In a similar fashion it can be shown that the series for the velocity components obtained from $\nabla\phi$ converges like $\Sigma|a_n|$. It follows that (2.11) does indeed represent a trapped mode when (2.25) is satisfied.

We can seek a trapped mode solution which is both odd in y and in x by expanding in terms of multipoles also odd in y and x. The resulting homogeneous infinite system differs from (2.13), (2.14), (2.10) in that $2n+1$, $2m+1$ are replaced by $2n$, $2m$ respectively and a factor of $\frac{1}{2}\epsilon_m$ ($\epsilon_0 = 1, \epsilon_m = 2, m = 1, 2, \ldots$) must be included in the right-hand side of (2.10). If we attempt to repeat the preceding argument in this case to prove the existence of a trapped mode in the limit $ka \to 0$, $kd \to \frac{1}{2}\pi$ we find that the construction fails.

Condition (2.15) ensures that the truncated determinant Δ_N converges uniformly to the limit Δ_∞ so that trapped modes can be obtained numerically by finding the zeros of Δ_N as N increases. A description of this procedure and the results obtained is given in the next section.

3. Numerical results

The computation of the integrals in (2.10) and the Bessel functions in (2.14) for different ka, kd is straightforward enabling Δ_N to be determined from (2.16) using a standard library routine. However, it was found that the value of N required to obtain a given accuracy in Δ_N was strongly dependent on the value of a/d. It is possible to reduce this dependence slightly by normalizing Δ_N by dividing the mth row by $1 + A_{mm}$ ($m = 0, 1, 2, \ldots$) as in the case of the rectangular block considered by Evans & Linton (1991). It should be noted, however, that whereas this was necessary to ensure numerical convergence of Δ_N as $N \to \infty$ in that case, here the convergence is already guaranteed.

Figure 1 shows curves of Δ_N plotted against kd for five values of a/d together with the value of N used in the calculation which ensures three significant figures of accuracy in the results. All the curves exhibit the same qualitative behaviour. As $kd \to 0$, Δ_N tends to a positive constant, whilst as $kd \to \frac{1}{2}\pi$, $\Delta_N \to -\infty$ and Δ_N has precisely one zero corresponding to the trapped mode frequency. An examination of the definition of A_{mn} given by (2.10) shows that with a/d fixed, as $kd \to 0$ the infinite integral is dominant whereas as $kd \to \frac{1}{2}\pi$ the finite integral is the dominant one. The results in the figure show that as $kd \to 0$, a/d fixed, the growth of the infinite integral is just balanced by the behaviour of $J'_{2m+1}(ka)/Y'_{2n+1}(ka)$ leading to a constant positive value for the determinant whereas in the limit $kd \to \frac{1}{2}\pi$ the Bessel functions do not vanish and the singular behaviour of the finite integral given by (B 5) results in the determinant tending to negative infinity.

Similar computations can be carried out for the infinite system arising from an attempt to construct trapped modes which are antisymmetric about $x = 0$. The same remarks concerning the qualitative behaviour in the $kd \to 0$ limit apply in this case and again as $kd \to \frac{1}{2}\pi$, Δ_N becomes infinite but since $J'_0(ka)/Y'_0(ka)$ is of opposite sign

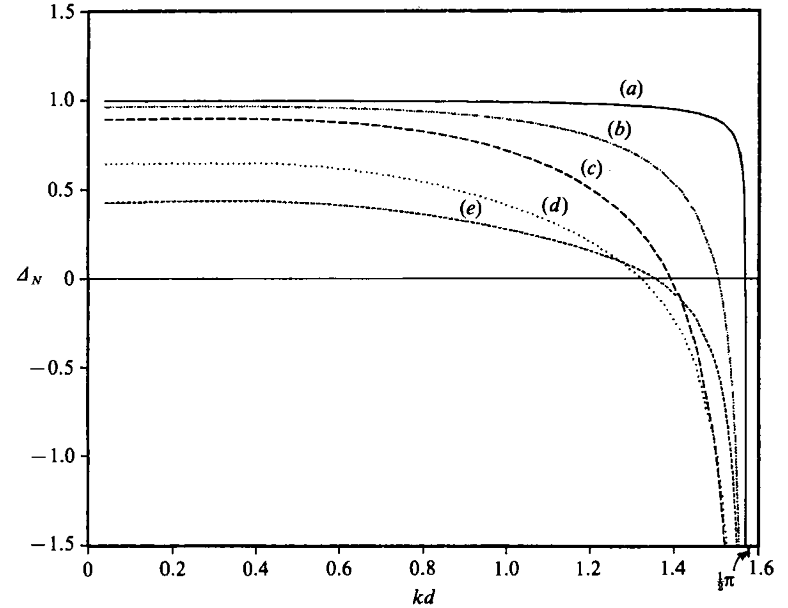

FIGURE 1. Variation of the determinant Δ_N of the system (2.26) with kd for different values of a/d, the ratio of cylinder radius to channel half-width. The value of N ensures three significant figure accuracy; zeros of Δ_N correspond to trapped mode frequencies. (a) $a/d = 0.1$, $N = 1$. (b) $a/d = 0.3$, $N = 2$. (c) $a/d = 0.5$, $N = 4$. (d) $a/d = 0.5$, $N = 12$. (e) $a/d = 0.9$, $N = 16$.

to $J_1'(ka)/Y_1'(ka)$, $\Delta_N \rightarrow +\infty$ and computations indicate that Δ_N has no zeros and hence there can be no trapped modes. This is consistent with the numerical results for the rectangular block obtained by Evans & Linton (1991), where it appeared that there were no trapped modes, antisymmetric about $x = 0$, unless $a/d > 1$ where a was the half-length of the block along the channel.

The values that are obtained for the trapped mode frequencies in the symmetric case are plotted in figure 2 together with the values obtained from the approximate formula

$$kd \sim \tfrac{1}{2}\pi(1-\tfrac{1}{8}\pi^4(a/d)^4), \tag{3.1}$$

which is a direct consequence of (2.26), valid for $a/d \rightarrow 0$, $kd \rightarrow \frac{1}{2}\pi$. The curves show that this formula is only applicable for a/d less than about 0.2, however computations confirm that (3.1) is extremely accurate for $a/d \leqslant 0.1$.

Just as for the rectangular block considered by Evans & Linton (1991), having similar dimensions to the circle considered here, figure 2 shows that there is little change in the trapped mode parameter kd as a/d varies, the minimum value being $kd \approx 1.32$. As $a/d \rightarrow 1$, $kd \sim 1.4$ in contrast to the block case where $kd \sim \frac{1}{2}\pi$. This latter result is explained in terms of the geometry of this limit since, as the block cuts off the entire channel the fundamental standing wave for the channel, having $kd = \frac{1}{2}\pi$, is approached. In contrast, when the circle occupies the entire width of the waveguide so that $a/d = 1$, there would appear to be a single non-trivial trapped mode with $kd < \frac{1}{2}\pi$. This is also in contrast to the case of the trapped surface wave modes above a horizontal submerged circular cylinder considered by Ursell (1951) and McIver & Evans (1985). In this case as $a/d \rightarrow 1$ and the cylinder approached the surface, numerical calculations suggested that the number of trapped modes increased indefinitely. This behaviour could be explained as follows. The limit is

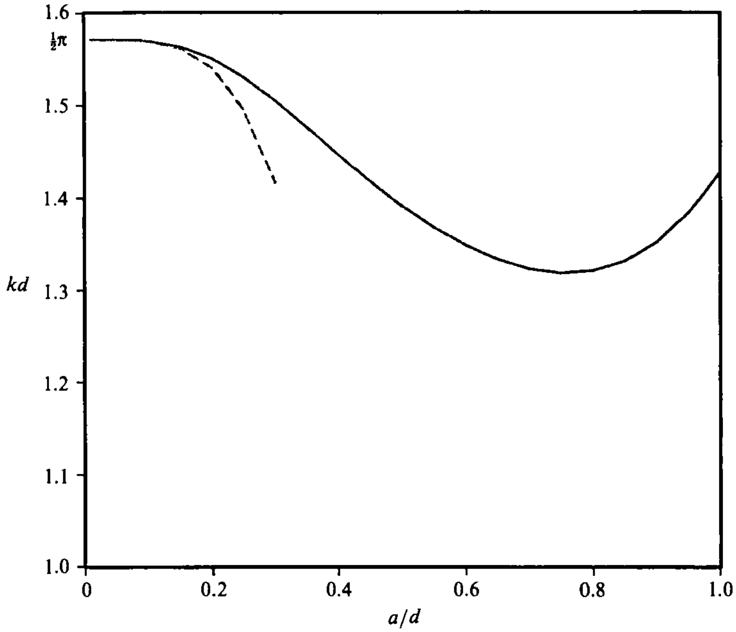

FIGURE 2. Variation of the trapped mode frequency parameter kd with a/d, the ratio of cylinder radius to channel half-width. The dotted line is the approximate relation (3.1) for small a/d derived from (2.26).

similar to edge waves over a gently sloping curved beach and it is known that the number of edge waves over a uniformly sloping beach increases as the beach angle tends to zero (Ursell 1952). In the present case the situation is different since it is the antisymmetry across the boundary $y = 0$ that guarantees a cutoff frequency and the possibility of trapped modes rather than the free-surface condition on $y = d$ as in the horizontal cylinder case.

This conclusion about the existence of a trapped mode in the case $a = d$ is not inconsistent with the proof given in Appendix B that the infinite system converges when $a/d < 1$ since the conditions for convergence are only sufficient. It is not clear how the bounds leading to (B 8) could be improved to include $a = d$.

4. Conclusion

It has been proved that there exists a trapped mode, or a local oscillation of well-defined frequency in the vicinity of a vertical fixed rigid circular cylinder symmetrically placed in a two-dimensional waveguide represented by two parallel lines enclosing a symmetrically-placed circle, or, equivalently, in an open water channel. The frequency of the oscillation is below the fundamental cutoff frequency for the channel or guide and depends upon the ratio of circle radius to guide width. The oscillation is antisymmetric about the centreline of the guide and symmetric about a vertical line perpendicular to the parallel walls. No trapped mode antisymmetric about both lines was found. An explicit relation between the radius of the circle and the trapped mode frequency parameter kd was found when the radius was small and the trapped mode frequency approached the cutoff frequency for the guide. The present work, by providing a rigorous proof of the existence of acoustic trapped modes in this particular case, extends our knowledge and understanding of these modes, complementing recent numerical work on vertical rectangular cylinders by Evans & Linton (1991) and approximate methods for vertical small cylinders of arbitrary cross-section by McIver (1991) or thin bodies by Evans & McIver (1991). The possibility of trapped modes over totally submerged horizontal cylinders or over sloping beaches has been known for some time and oceanographers have observed these modes, more commonly called edge waves, over continental shelves. A good description is given by LeBlond and Mysak (1978). The proof of the existence of trapped modes in an acoustic waveguide given here is restricted to small circles in parallel waveguides. However, the numerical evidence for acoustic trapped modes in a wide range of situations is overwhelming and there can be little doubt of their existence in more general waveguide problems. A discussion of such problems is given in Evans & Linton (1991).

It is straightforward to repeat the construction used here for the condition $\phi = 0$ on the cylinder and to seek possible trapped modes. However, such modes do not appear to exist in this case.

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Appendix A. Derivation and expansion of multipoles

An integral representation of the Hankel function $H_n^{(1)}$ due to Sommerfeld, given by Erdélyi *et al.* (1953, vol. 2, p. 20, equation (20)) is

$$H_n^{(1)}(kr) = -\frac{1}{\pi}\int_{a+\mathrm{i}\infty}^{b-\mathrm{i}\infty} \mathrm{e}^{\mathrm{i}kr\cos\alpha}\,\mathrm{e}^{\mathrm{i}n(\alpha-\frac{1}{2}\pi)}\,\mathrm{d}\alpha \quad (-\pi < a < 0, \quad 0 < b < \pi).$$

The change of variable $\alpha = \mathrm{i}(v+\theta)$, and the choice of $a+\theta = 0$, $b+\theta = \pi$ results in

$$H_n^{(1)}(kr)\,\mathrm{i}^n\,\mathrm{e}^{\mathrm{i}n\theta} = -\frac{\mathrm{i}}{\pi}\int_{-\infty}^{\infty+\mathrm{i}\pi} \mathrm{e}^{\mathrm{i}kx\cosh v}\,\mathrm{e}^{ky\sinh v}\,\mathrm{e}^{nv}\,\mathrm{d}v \quad \text{for } 0 < \theta < \pi,$$

where $x = r\cos\theta$, $y = r\sin\theta$, and where the contour is along the negative real axis to the origin, up the imaginary axis to $\mathrm{i}\pi$ and then along $v = \mathrm{i}\pi + s$, $s > 0$. It follows that

$$H_{2n+1}^{(1)}(kr)\sin(2n+1)\theta = \frac{-\mathrm{i}(-1)^n}{\pi}\int_{-\infty}^{\infty+\mathrm{i}\pi} \exp\gamma y\cos(kx\cosh v)\,\mathrm{e}^{-(2n+1)v}\,\mathrm{d}v, \quad \text{(A 1)}$$

where $\gamma = k\sinh v$, is an integral representation of the fundamental singular multipole satisfying Helmholtz's equation which is even in x and odd in y.

We seek to add an expression to (A 1) to satisfy the condition on $y = d$. We shall choose a function odd in y which provides an extension into $y < 0$ and ensures the condition on $y = -d$ is also satisfied. To this end we note that

$$\frac{\partial}{\partial y}\{\mathrm{e}^{\gamma y} + A(v)\sinh\gamma y\} = 0 \quad \text{on } y = d,$$

if $$A(v) = -\mathrm{e}^{\gamma d}/\cosh\gamma d.$$

It follows that the required multipole is

$$\phi_{2n+1}(kr,\theta) = H^{(1)}_{2n+1}(kr)\sin(2n+1)\theta + \frac{\mathrm{i}(-1)^n}{\pi}\int_{-\infty}^{\infty+\mathrm{i}\pi} \frac{\exp\gamma d}{\cosh\gamma d}\sinh\gamma y\cos(kx\cosh v)\,\mathrm{e}^{-(2n+1)v}\,\mathrm{d}v, \quad (\mathrm{A}\ 2)$$

$$= \frac{-\mathrm{i}(-1)^n}{\pi}\int_{-\infty}^{\infty+\mathrm{i}\pi} \frac{\cosh\gamma(d-y)}{\cosh\gamma d}\cos(kx\cosh v)\,\mathrm{e}^{-(2n+1)v}\,\mathrm{d}v, \quad (\mathrm{A}\ 3)$$

where (A 1) has been used in (A 2) to obtain (A 3).

The contour integral in (A 3) can be expressed in three parts in an obvious notation:

$$\int_{-\infty}^{0} + \int_{0}^{\mathrm{i}\pi} + \int_{\mathrm{i}\pi}^{\mathrm{i}\pi+\infty}.$$

We write $v = -v$ in the first, $v = \mathrm{i}u + \mathrm{i}\tfrac{1}{2}\pi$ in the second and $v = \mathrm{i}\pi + v$ in the third to obtain

$$\phi_{2n+1}(kr,\theta) = H^{(1)}_{2n+1}(kr)\sin(2n+1)\theta - \frac{2\mathrm{i}(-1)^n}{\pi}\int_0^\infty \frac{\mathrm{e}^{-\gamma d}}{\cosh\gamma d}\sinh\gamma y\cos(kx\cosh v)\sinh(2n+1)v\,\mathrm{d}v - \frac{2}{\pi}\int_0^{\frac{1}{2}\pi}\frac{\mathrm{e}^{\mathrm{i}\beta d}}{\cos\beta d}\sin\beta y\cos(kx\sin u)\cos(2n+1)u\,\mathrm{d}u, \quad (\mathrm{A}\ 4)$$

$$= -\frac{2\mathrm{i}(-1)^n}{\pi}\int_0^\infty \frac{\cosh\gamma(d-y)}{\cosh\gamma d}\cos(kx\cosh v)\sinh(2n+1)v\,\mathrm{d}v - \frac{2\mathrm{i}}{\pi}\int_0^{\frac{1}{2}\pi}\frac{\cos\beta(d-y)}{\cos\beta d}\cos(kx\sin u)\cos(2n+1)u\,\mathrm{d}u, \quad (\mathrm{A}\ 5)$$

where $\beta = k\cos u$ and both integrals are real in (A 5).

The expression (A 4) clearly vanishes when $y = 0$. This is not so clear from the alternative expression (A 5) but what is made evident from this form is that $\mathrm{Re}\,\phi_{2n+1}(kr,\theta) = 0$. Also since $kd < \tfrac{1}{2}\pi$, $\cos\beta d$ does not vanish for any real u in $(0, \tfrac{1}{2}\pi)$, and, from the Riemann–Lebesgue lemma

$$\phi_{2n+1} \to 0, \quad |x| \to \infty, \quad 0 < y < d. \quad (\mathrm{A}\ 6)$$

Now $$\sinh\gamma y\cos(kx\cosh v) = 2\sum_{n=0}^{\infty}(-1)^n J_{2n+1}(kr)\sin(2n+1)v\sinh(2n+1)\theta, \quad (\mathrm{A}\ 7)$$

a result derived from the identity (Watson 1966, p. 15)

$$\exp\{\tfrac{1}{2}z(\tau-\tau^{-1})\} = J_0(z) + \sum_{n=1}^{\infty}(\tau^n + (-1)^n\tau^{-n})J_n(z).$$

We substitute (A 7) into the imaginary part of (A 2) to obtain

$$\psi_{2n+1}(kr,\theta) = Y_{2n+1}(kr)\sin(2n+1)\theta + \sum_{m=0}^{\infty}A_{mn}J_{2m+1}(kr)\sin(2m+1)\theta, \quad (\mathrm{A}\ 8)$$

where $$A_{mn} = \frac{2(-1)^{m+n}}{\pi}\,\mathrm{Re}\int_{-\infty}^{\infty+\mathrm{i}\pi} \frac{\exp(\gamma d)}{\cosh\gamma d}\sinh(2m+1)v\,\mathrm{e}^{-(2n+1)v}\,\mathrm{d}v. \tag{A 9}$$

We can split this contour integral into three parts as before to obtain

$$A_{mn} = \frac{-4}{\pi}(-1)^{m+n}\int_0^\infty \frac{\mathrm{e}^{-\gamma d}}{\cosh\gamma d}\sinh(2n+1)v\sinh(2m+1)v\,\mathrm{d}v$$
$$-\frac{4}{\pi}\int_0^{\frac{1}{2}\pi}\tan\beta d\cos(2n+1)u\cos(2m+1)u\,\mathrm{d}u,$$

where $$\gamma = k\sinh v, \quad \beta = k\cos u. \tag{A 10}$$

Appendix B. Proof of convergence of the infinite system

We shall show that

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}|B_{mn}| < \infty \quad \text{for } 0 < ka < kd < \tfrac{1}{2}\pi, \coth\chi < M(ka)^{-2},$$
$$M \text{ a constant}, \quad kd\cosh\chi = \tfrac{1}{2}\pi,$$

where $$B_{mn} = A_{mn}\,J'_{2m+1}(ka)/Y'_{2n+1}(ka),$$

and $$A_{mn} = -(4/\pi)\left((-1)^{m+n}I_1 + I_2\right),$$

where $$I_1 = \int_0^\infty \frac{\mathrm{e}^{-\gamma d}\sinh(2n+1)v\sinh(2m+1)v}{\cosh(\gamma d)}\,\mathrm{d}v \qquad (\gamma = k\sinh v),$$

$$I_2 = \int_0^{\frac{1}{2}\pi}\tan(\beta d)\cos(2n+1)u\cos(2m+1)u\,\mathrm{d}u \quad (\beta = k\cos u).$$

We shall need the following rough bounds on Bessel functions which can be derived from their series expansions and asymptotic expansions for large order and fixed argument. See, for example, Abramowitz & Stegun (1965) pp. 375, 362, and 364.

$$|K_n(x)| < M_1(n-1)!/(\tfrac{1}{2}x)^n, \tag{B 1}$$
$$|J_n(x)| < M_2(\tfrac{1}{2}x)^{n-1}/(n-1)!, \tag{B 2}$$
and $$|Y'_n(x)| > M_3\,n!/(\tfrac{1}{2}x)^{n+1}, \tag{B 3}$$

for $n > N$, say where the $M_i\,(i = 1,2,3)$ are independent of x, and $0 < x \leqslant \frac{1}{2}\pi$ which is sufficient for our purposes.

In estimating I_1 we shall make use of elementary inequalities involving hyperbolic functions. Thus

$$|I_1| < \int_0^\infty \mathrm{e}^{-2kd\sin v}\cosh rv\,\mathrm{d}v \quad (r = 2n+2m+2)$$
$$< M_4\int_0^\infty \mathrm{e}^{-2kd\cosh v}\cosh rv\,\mathrm{d}v,$$
$$= M_4K_r(2kd) \qquad \text{(Watson 1966, p. 181)}$$
$$< M_5(2n+2m+1)!/(kd)^{2n+2m+2}, \tag{B 4}$$

from (B 1).

Now $$2I_2 = \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\tan\beta d\cos(2n+1)u\cos(2m+1)u\,\mathrm{d}u \quad (\beta = k\cos u),$$

and the integrand has poles at $u = \pm i\chi$, where $kd \cosh\chi = \frac{1}{2}\pi$. Choose a positive number δ, such that $0 < \chi \leqslant \frac{1}{2}\delta$ and deform the contour of integration upwards over the pole at $u = i\chi$, into the rectangular path

$$C: u = -\tfrac{1}{2}\pi + iv \quad (0 \leqslant v \leqslant \delta); \quad u = v + i\delta(-\tfrac{1}{2}\pi \leqslant v \leqslant \tfrac{1}{2}\pi; \quad u = \tfrac{1}{2}\pi + iv \quad (\delta \geqslant v \geqslant 0).$$

Then
$$2I_2 = \frac{2\pi}{kd \sinh\chi} \cosh(2n+1)\chi \cosh(2m+1)\chi + \int_c \tag{B 5}$$

where it is easily shown that $|\int_c| < 2M_6(\delta) \exp(2n+2m+2)\delta$, a bound which is uniform in δ as $\chi \to 0$ since $0 < \chi < \frac{1}{2}\delta$. Thus

$$|I_2| < M_7(\delta) \coth\chi \exp(2n+2m+2)\delta, \tag{B 6}$$

and
$$\left|\frac{J'_{2m+1}(ka)}{Y'_{2n+1}(ka)}\right| < \frac{M_8(\frac{1}{2}ka)^{2m+2n+2}}{2m!(2n+1)!} \quad (n, m > N), \tag{B 7}$$

from (B 2), (B 3).

It follows that

$$\sum_{N+1}^{\infty} \sum_{N+1}^{\infty} |B_{mn}| < (ka)^2 M_9 S_1^2 \coth\chi + M_{10} S_2, \tag{B 8}$$

where
$$S_1 = \sum_{n=0}^{\infty} \frac{(\frac{1}{2}ka \exp\delta)^{2n}}{2n!} < \infty,$$

and
$$S_2 = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left(\frac{a}{2d}\right)^{2n+2m+2} \frac{(2n+2m+1)!}{2m!(2n+1)!}$$

$$= \sum_{r=0}^{\infty} \left(\frac{a}{2d}\right)^{2r+2} \sum_{m+n=r} \frac{(2r+1)!}{2m!(2n+1)!},$$

after summing diagonally. But

$$\sum_{m+n=r} \frac{(2r+1)!}{2m!(2n+1)!} = \sum_{m=0}^{r} \frac{(2r+1)!}{2m!(2r-2m+1)!} = 2^{2r}$$

where the last series has been summed as $(1+x)^{2r+1} + (1-x)^{2r+1}$ with $x = 1$. Thus

$$S_2 = \frac{a^2}{4(d^2-a^2)}, \tag{B 9}$$

provided $a/d < 1$, and hence

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} |B_{mn}| < \text{constant},$$

for $0 < ka < kd < \frac{1}{2}\pi$, and $\coth\chi < M(ka)^{-2}$, from (B 8).

Appendix C. Behaviour of B_{mn} as $ka \to 0$, $kd \to \frac{1}{2}\pi$

We have

$$B_{mn} = A_{mn} \frac{J'_{2m+1}(ka)}{Y'_{2n+1}(ka)}, \tag{C 1}$$

where A_{mn} is given by the two real integrals in (A 10) which depend only upon kd.

Now

$$\frac{J'_{2m+1}(ka)}{Y'_{2n+1}(ka)} \sim \frac{\pi(2m+1)}{(2n+1)\,(2n+1)\,!}(\tfrac{1}{2}ka)^{2n+2m+2} \quad (ka \to 0), \tag{C 2}$$

from the series expansions of the Bessel function in, for example, Watson (1966).

It remains to consider A_{mn} as $kd \to \frac{1}{2}\pi$. From (B 4) the infinite integral in (A 10) is clearly bounded as $kd \to \frac{1}{2}\pi$, n, m fixed, so it remains to consider the second integral in (A 10) as $kd \to \frac{1}{2}\pi$, or $\chi \to 0$.

From (B 5) we see

$$I_2 \sim 2 \coth \chi \quad \text{as } \chi \to 0, \tag{C 3}$$

and so

$$A_{mn} \sim -\frac{8}{\pi} \coth \chi \quad (\chi \to 0), \tag{C 4}$$

and finally

$$B_{mn} \sim \frac{-8(2m+1)}{(2n+1)\,(2n+1)\,!}(\tfrac{1}{2}ka)^{2n+2m+2} \coth \chi \quad (ka \to 0, \chi \to 0). \tag{C 5}$$

Appendix D. Convergence of the trapped mode expansion

We have, from (2 9) and (2 11)

$$\begin{aligned}\phi(a,\theta) &= \sum_{n=0}^{\infty} \frac{a_n}{kY'_{2n+1}(ka)}\left\{Y_{2n+1}(ka)\sin(2n+1)\,\theta + \sum_{m=0}^{\infty} A_{mn}\,J_{2m+1}(ka)\sin(2m+1)\,\theta\right\} \\ &= -\sum_{m=0}^{\infty} \frac{2}{\pi k^2 a}\frac{a_m \sin(2m+1)\,\theta}{Y'_{2m+1}(ka)\,J'_{2m+1}(ka)},\end{aligned} \tag{D 1}$$

where (2 12) and the identity

$$J_{2m+1}\,Y'_{2m+1} - J'_{2m+1}\,Y_{2m+1} = 2/\pi ka$$

(Erdélyi *et al.* 1953, vol. 2, p. 79) have been used.

But $J'_{2m+1}\,Y'_{2m+1} \sim (2m+1)/\pi(ka)^2$, $m \to \infty$, ka fixed, which, since $\Sigma|a_m| < \infty$, is sufficient to show that the series in (D 1) converges and is clearly non-zero, so that the trapped mode potential does not vanish identically.

We need to show that the series expression (2.11) is convergent for $r \geqslant a$. Consider the expression (A 4) for ϕ_{2n+1} whose imaginary part is ψ_{2n+1}. Denote the infinite integral by I_5. For $v \in (0, \infty)$ we have, with $\gamma = k \sinh v$

$$\mathrm{e}^{-\gamma d}/\cosh \gamma d < 2\,\mathrm{e}^{-2\gamma d}, \quad \sinh \gamma y < \mathrm{e}^{\gamma d},$$

and

$$\mathrm{e}^{-\gamma d} = \mathrm{e}^{-kd\cosh v}\,\mathrm{e}^{kd\,\mathrm{e}^{-v}} < \mathrm{e}^{kd}\,\mathrm{e}^{-kd\cosh v}.$$

Hence, from (B 1),

$$\begin{aligned}|I_5| &< 2\,\mathrm{e}^{kd}\int_0^{\infty} \mathrm{e}^{-kd\cosh v}\cosh(2n+1)\,v\,\mathrm{d}v = 2\,\mathrm{e}^{kd}K_{2n+1}(kd) \\ &< 2M_1\,\mathrm{e}^{kd}\,2n\,!/(kd/2)^{2n+1} \quad \text{for } N > n.\end{aligned} \tag{D 2}$$

Again, denoting the imaginary part of the second integral in (A 4) by I_6 we have, with $u \in (0, \frac{1}{2}\pi)$, $\beta = k\cos u$, $kd < \frac{1}{2}\pi$, $\tan \beta d < \tan kd$, and so,

$$|I_6| < \tan kd. \tag{D 3}$$

Finally (Abramowitz & Stegun 1965, p. 365),

$$|H^{(1)}_{2n+1}(kr)| \sim \frac{2^{2n+1}2n!}{\pi(kr)^{2n+1}} \leqslant \frac{2^{2n+1}2n!}{\pi(ka)^{2n+1}} \quad (r \geqslant a), \tag{D 4}$$

for n large, kr fixed.

Combining the inequalities (D 2), (D 3) and (D 4) with (B 3) shows that for $a \leqslant d$ and $r \geqslant a$ the series for $\phi(r, \theta)$ given by (2.11) converges at least as fast as $\Sigma|a_n|/(2n+1)$ whilst the series for $\nabla\phi(r, \theta)$ converges at least as fast as $\Sigma|a_n|$.

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